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Local existence of symmetric spinor potentials for symmetric (3,1)-spinors in Einstein space–times

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Abstract

We investigate the possibility of existence of a symmetric potential $H_{ABA'B'} = H_{(AB)(A'B')}$ for a symmetric (3,1)-spinor $L_{ABCA'}$, e.g., a Lanczos potential of the Weyl spinor, as defined by the equation $L_{ABCA'} = \nabla_{(A}{}^{B'} H_{BC)A'B'}$. We prove that in all Einstein space–times such a symmetric potential $H_{ABA'B'}$ exists. Potentials of this type have been found earlier in investigations of some very special spinors in restricted classes of space–times. A tensor version of this result is also given. We apply similar ideas and results by Illge to Maxwell's equations in a curved space–time.

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1. Introduction

More than 30 years ago Lanczos [20] proposed a first order potential for the Weyl tensor. However, in 1983 Bampi and Caviglia [7] showed that Lanczos' original proof was flawed and supplied a rigorous but complicated proof of local existence for four-dimensional analytic spaces, independent of signature. Illge [18] has supplied a more conventional proof of existence (by means of a Cauchy problem) in spinor notation that, in its full generality, does not seem to generalize in an obvious manner, to arbitrary signature. Moreover, it should be emphasized that Illge's work has highlighted the simple and natural structure of

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the Lanczos potential *in spinor notation*, and makes it clear that for work in space–times (four-dimensional C^∞ manifolds with Lorentz signature) the spinor formalism is much simpler than the tensor formalism. It should also be noted that in Lorentz signature the Lanczos potential satisfies a wave equation, and the well-posedness of the corresponding Cauchy problem enabled Illge to remove the assumption about analyticity in his proof.

It is important to note that the two existence proofs supplied by Bampi and Caviglia [7], and by Illge [18], respectively, do not directly concern the Weyl tensor/spinor C_{abcd}/Ψ_{ABCD}, but are valid for *any* tensor/spinor W_{abcd}/W_{ABCD} having the same algebraic symmetries as the Weyl tensor/spinor. Furthermore, Illge's work also discusses the existence of potentials for completely symmetric spinors with an arbitrary number of primed and unprimed indices; in general these potentials are not symmetric.

For a recent, very simple and direct proof of the existence of the Lanczos potential (see [6]). This proof introduces an asymmetric 'superpotential' $T_{ABCD} = T_{(ABC)D}$ such that $L_{ABCA'} = \nabla_{A'}{}^{D} T_{ABCD}$.

The results in this paper for four-dimensional space–times are given following the conventions of [25] in spinor notation, where the results are natural and the calculations comparatively simple; however, we also give the results in tensor notation for four-dimensional space–times. We remark that in general the Lanczos potential does not exist in dimensions higher than 4 [14].

Note that a spinor $S_{A_1\cdots A_n B'_1\cdots B'_m}$ having both primed and unprimed indices is said to be (*completely*) *symmetric* if it is symmetric over both types of indices, i.e.,

$$S_{A_1\cdots A_n B'_1\cdots B'_m} = S_{(A_1\cdots A_n)(B'_1\cdots B'_m)}.$$

In Section 2, we state Illge's theorem for the existence and uniqueness of a symmetric spinor potential $L_{A_1\cdots A_n A'}$ for the symmetric spinor $W_{AA_1\cdots A_n}$. We also state the analogous result for a spinor potential $L_{A_1\cdots A_n B'_1\cdots B'_m A'} (= L_{(A_1\cdots A_n)(B'_1\cdots B'_m)A'})$ for the completely symmetric spinor $W_{AA_1\cdots A_n B'_1\cdots B'_m}$. In addition, we quote the corresponding Lanczos wave equation for $L_{ABCA'}$ and $L_{A_1\cdots A_n B'_1\cdots B'_m A'}$ showing how in the latter case an *algebraic* constraint arises in general, if we try to demand a *completely symmetric* $L_{A_1\cdots A_n B'_1\cdots B'_m A'}$.

As noted above, Illge has shown the existence of (asymmetric) potentials for completely symmetric spinors with an arbitrary number of primed and unprimed indices. Thus, a Lanczos potential $L_{ABCA'}$ of some symmetric spinor W_{ABCD} itself has spinor potentials. One example is the spinor T_{ABCD} referred to above, but there are reasons why we are more interested in having a *symmetric* potential of the type $H_{ABA'B'} = H_{(AB)(A'B')}$ (see, e.g., [1,3,5,9,27,28]). Although such a potential does not exist in all space–times, we demonstrate in Section 3 that it does exist in all Einstein space–times. In order to obtain a unique solution to the problem, we will supplement the defining equation for $H_{ABA'B'}$ with certain other conditions and use a technique which is similar in structure to Illge's proof for the existence of $L_{ABCA'}$. As a result our proof of this result will lack the simplicity of the existence proof for $L_{ABCA'}$ given in [6]. A tensor version is also given.

In Section 4, we will look in more detail at the important application to electromagnetism in curved space. We do this in order to see how the results in the first sections relate to more familiar results on potentials such as Poincare's lemma, and what simplifications can be achieved due to the simpler index structure of the electromagnetic spinor, and due to Maxwell's equations.

In Section 5, we discuss how the results in this paper links up with existing results and applications.

2. Preliminaries

Let M be a space–time (i.e., a real, C^∞, four-dimensional manifold with a metric of signature $(+---)$). For simplicity, we will restrict ourselves to tensor- and spinor fields of class C^∞, but note that the results given could be generalized to tensor- and spinor fields of lesser regularity by using theorems on hyperbolic systems where the fields are only assumed to be in some Sobolev space, instead of the theorems used here. For definitions of the Levi-Civita connection, the curvature spinors, etc., we will follow the conventions in [25]. Also note that all indices (both tensor- and spinor-indices) occurring in this paper are abstract indices [25].

Illge [18] has shown that given any symmetric spinors $W_{ABCD} = W_{(ABCD)}$, $F_{BC} = F_{(BC)}$ there exists (locally) a symmetric spinor $L_{ABCA'} = L_{(ABC)A'}$ such that

$$W_{ABCD} = 2\nabla_{(A}{}^{A'} L_{BCD)A'}, \qquad F_{BC} = \nabla^{AA'} L_{ABCA'}.$$

Such a spinor $L_{ABCA'}$ is said to be a Lanczos (spinor) potential of W_{ABCD}. The spinor F_{BC} is called the differential gauge of $L_{ABCA'}$. When $F_{BC} = 0$, the Lanczos potential is said to be in Lanczos differential gauge. Of particular interest is the case $W_{ABCD} = \Psi_{ABCD}$, i.e., Lanczos potentials of the Weyl curvature spinor. In this case the first of the above equations is called the Weyl–Lanczos equation. These Lanczos potentials are spinor analogues of the Lanczos *tensor* potentials, originally investigated in [20]. For an extensive account of the Lanczos potential and its properties see [1,13].

One of the most remarkable results concerning Lanczos potentials is Illge's wave equation [18]. Suppose $L_{ABCA'}$ is a Lanczos potential of W_{ABCD} in the differential gauge F_{BC}. Then, $L_{ABCA'}$ satisfies the following linear wave equation:

$$\Box L_{ABCA'} + 6\Phi_{A'B'(A}{}^{D} L_{BC)D}{}^{B'} + 6\Lambda L_{ABCA'} + \nabla^{D}_{A'} W_{ABCD} - \tfrac{3}{2}\nabla_{A'(A} F_{BC)} = 0.$$

Now, if W_{ABCD} is actually the Weyl spinor Ψ_{ABCD}, if the space–time is vacuum and if $L_{ABCA'}$ is in Lanczos differential gauge, we obtain the remarkably simple equation

$$\Box L_{ABCA'} = 0.$$

By letting L_{abc} be the tensor equivalent of the hermitian spinor

$$L_{abc} = L_{ABCC'}\varepsilon_{A'B'} + \bar{L}_{A'B'C'C}\varepsilon_{AB},$$

the tensor L_{abc} has the symmetries $L_{abc} = L_{[ab]c}$, $L_{[abc]} = 0$, $L_{ab}{}^{b} = 0$. This last symmetry was originally thought of as a gauge condition called the Lanczos algebraic gauge; however, because of the spinor correspondence we choose to include this symmetry in the definition of the Lanczos potential. As we shall see below, it also gives us a comparatively simple form of the tensor equation corresponding to the Weyl–Lanczos equation.

We can now define a Lanczos *tensor* potential of the Weyl tensor C_{abcd}, or indeed of any tensor W_{abcd} having the same algebraic symmetries as the Weyl tensor, by translating the Weyl–Lanczos equation into tensor formalism. We obtain the Weyl–Lanczos *tensor* equation which reads

$$W_{abcd} = L_{ab[c;d]} + L_{cd[a;b]} - {}^*L^*_{ab[c;d]} - {}^*L^*_{cd[a;b]}, \tag{1}$$

where W_{abcd} has the same algebraic symmetries as the Weyl tensor. This is the original definition of the Lanczos potential given in [20]. By differentiating the Weyl–Lanczos tensor equation and using the Bianchi identities and the commutators we obtain a wave equation, similar to Illge's spinor wave equation. It is

$$\Box L_{abc} - 2L^{def} g_{c[a} C_{b]def} + 2L_{[a}{}^{de} C_{b]edc} + \tfrac{1}{2} L^{de}{}_c C_{deab} = 0$$

in vacuum, Lanczos differential gauge and $W_{abcd} = C_{abcd}$. It is interesting to note that it is much more difficult to calculate this tensor wave equation than the corresponding spinor one, and it also appears at first to have a much more complicated structure as it seems to include an expression involving products of the Weyl tensor and the Lanczos potential explicitly. However, it was shown in [12], by some very simple manipulations using Hodge duals, that this expression vanishes identically in four, and only in four dimensions, irrespective of metric signature, so the tensor wave equation simplifies to

$$\Box L_{abc} = 0$$

in agreement with the spinor equation. The same result can also be proved using a dimensionally dependent identity by Lovelock [22]. This technique is illustrated in [4,17]. An alternative proof, valid only in Lorentz signature is given in [11], using spinor techniques.

An interesting result regarding this wave equation was proved by Edgar and Höglund [13]. They showed that in a vacuum space–time of 'sufficient generality' (see [13] or [8]) a spinor $L_{ABCA'}$ in Lanczos differential gauge $\nabla^{AA'} L_{ABCA'} = 0$ is a constant multiple of a Lanczos potential of the Weyl spinor if and only if $\Box L_{ABCA'} = 0$. Hence, in this particular case, Illge's wave equation is actually a sufficient condition for $L_{ABCA'}$ to be a constant multiple of a Lanczos potential of the Weyl spinor.

Illge's theorem in [18] is actually more general than we have quoted above. Illge proves the existence of a potential similar to the one mentioned above, for the case when the symmetric spinor W has an arbitrary number of indices. For easy reference we include the complete theorem of Illge in this section, together with a generalization also mentioned in [18].

Theorem 2.1. *Let symmetric spinor fields* $W_{AA_1\cdots A_n}$, $F_{A_2\cdots A_n}$, *a space-like past-compact hypersurface* Σ *of class* C^∞ *and a symmetric spinor field* $\mathring{L}_{A_1\cdots A_n A'}$ *defined only on* Σ [1]

[1] From now on, a circle above a spinor field, i.e., $\mathring{L}$ will always mean that the spinor field is defined only on Σ.

be given. Then there exists a neighbourhood U (*called a causal neighbourhood* [15]) *of* Σ *in which the equations*

$$W_{AA_1\cdots A_n} = 2\nabla_{(A}{}^{A'} L_{A_1\cdots A_n)A'}, \qquad F_{A_2\cdots A_n} = \nabla^{A_1A'} L_{A_1A_2\cdots A_nA'}$$

have a unique symmetric solution $L_{A_1\cdots A_nA'}$ *satisfying* $L|_\Sigma = \mathring{\mathrm{L}}$.

We note that a spinor W_{ABCD} in general has many Lanczos potentials in each differential gauge F_{BC}.

Following Illge [18], we attempt to generalize this theorem to symmetric spinors with both primed and unprimed indices. Let $W_{AA_1\cdots A_nB'_1\cdots B'_m}$ and $F_{A_2\cdots A_nB'_1\cdots B'_m}$ be completely symmetric spinors. We then look for a spinor $L_{A_1\cdots A_nB'_1\cdots B'_mA'}$ so that

$$W_{AA_1\cdots A_nB'_1\cdots B'_m} = 2\nabla_{(A}{}^{A'} L_{A_1\cdots A_n)B'_1\cdots B'_mA'},$$
$$F_{A_2\cdots A_nB'_1\cdots B'_m} = \nabla^{A_1A'} L_{A_1A_2\cdots A_nB'_1\cdots B'_mA'}.$$

From this equation, we see that it is natural to require that L has the symmetry

$$L_{A_1\cdots A_nB'_1\cdots B'_mA'} = L_{(A_1\cdots A_n)(B'_1\cdots B'_m)A'}.$$

By combining the above two equations into one, differentiating and using the commutators, we arrive at a wave equation analogous to Illge's wave equation

$$\begin{aligned} 0 = {} & \Box L_{A_1\cdots A_nB'_1\cdots B'_mA'} - 2n\Phi_{B'A'D(A_1} L^D{}_{A_2\cdots A_n)B'_1\cdots B'_m}{}^{B'} \\ & -2m\bar{\Psi}_{A'B'C'(B'_1} L_{|A_1\cdots A_n|B'_2\cdots B'_m)}{}^{B'C'} + 2\Lambda(3L_{A_1\cdots A_nB'_1\cdots B'_mA'} \\ & -m(L_{A_1\cdots A_nA'(B'_1\cdots B'_m)} + \varepsilon_{A'(B'_1} L_{|A_1\cdots A_n|B'_2\cdots B'_m)C'}{}^{C'})) \\ & +\nabla_{A'}{}^{A} W_{AA_1\cdots A_nB'_1\cdots B'_m} - \frac{2n}{n+1}\nabla_{A'(A_1} F_{A_2\cdots A_n)B'_1\cdots B'_m}. \end{aligned} \tag{2}$$

Note that the corresponding Eq. (8) given in [18] contains a few misprints. Suppose, we first try to find a completely symmetric solution of this equation, i.e.,

$$L_{A_1\cdots A_nB'_1\cdots B'_mA'} = L_{(A_1\cdots A_n)(B'_1\cdots B'_mA')}.$$

Multiplying (2) by $\varepsilon^{A'B'_1}$ gives

$$\begin{aligned} 0 = {} & n\Phi_{A'B'D(A_1} L^D{}_{A_2\cdots A_n)B'_2\cdots B'_m}{}^{A'B'} + (m-1)\bar{\Psi}_{A'B'C'(B'_2} L_{|A_1\cdots A_n|B'_3\cdots B'_m)}{}^{A'B'C'} \\ & +\tfrac{1}{2}\nabla^{AA'} W_{AA_1\cdots A_nA'B'_2\cdots B'_m} - \frac{n}{n+1}\nabla_{(A_1}{}^{A'} F_{A_2\cdots A_n)A'B'_2\cdots B'_m}. \end{aligned} \tag{3}$$

Thus, we obtain not only a wave equation for L, but also an *algebraic constraint* on the potential L. Therefore we cannot, in general, find a completely symmetric potential for a spinor field with both primed and unprimed indices. However, we immediately see some cases where these constraints are automatically satisfied, e.g., when $n = 0, m = 1$ providing $\nabla^{AA'} W_{AA'} = 0$. Illge [18] proves that in this case a symmetric potential exists.

Also, if $m = 1$ and $\Phi_{ABA'B'} = 0$, then the potential vanishes from this constraint equation, and we are left with just an equation for the differential gauge F. If this equation can be solved, we might expect to find a potential for W. These ideas will be explored in detail in Section 3. On the other hand, if $n = 0$ and $\Psi_{ABCD} = 0$, we also see that the above equation is no longer a constraint on the potential itself.

So, in particular, we see that for $W_{AA_1\cdots A_nA'}$ the possibility of having a potential of type $L_{A_1\cdots A_nA'B'} = L_{(A_1\cdots A_n)(A'B')}$ in space–times with vanishing Ricci spinor, is not ruled out. The possibility of $W_{AB'_1\cdots B'_m}$ having a symmetric potential $L_{B'_1\cdots B'_mA'}$ in conformally flat space–times is not ruled out either.

Finally, we note that if we do not require complete symmetry of L, then no constraints occur, and Illge has proven the following generalization of Theorem 2.1 (see [18]).

Theorem 2.2. *Let symmetric spinor fields* $W_{AA_1\cdots A_nB'_1\cdots B'_m}$, $F_{A_2\cdots A_nB'_1\cdots B'_m}$, *a space-like past-compact hypersurface* Σ *of class* C^∞ *and a spinor field*

$$\mathring{L}_{A_1\cdots A_nB'_1\cdots B'_mA'} = \mathring{L}_{(A_1\cdots A_n)(B'_1\cdots B'_m)A'}$$

defined only on Σ *be given. Then, there exists a neighbourhood U* (*called a causal neighbourhood* [15]) *of* Σ *in which the equations*

$$W_{AA_1\cdots A_nB'_1\cdots B'_m} = 2\nabla_{(A}{}^{A'}L_{A_1\cdots A_n)B'_1\cdots B'_mA'},$$
$$F_{A_2\cdots A_nB'_1\cdots B'_m} = \nabla^{A_1A'}L_{A_1A_2\cdots A_nB'_1\cdots B'_mA'}$$

have a unique solution $L_{A_1\cdots A_nB'_1\cdots B'_mA'} = L_{(A_1\cdots A_n)(B'_1\cdots B'_m)A'}$ *that satisfies* $L|_\Sigma = \mathring{L}$.

In summary, Illge has shown that any symmetric spinor (in fact the symmetry condition is not necessary, although these are usually the spinors we are interested in) has a potential (actually two different potentials using Theorem 2.2 and the complex conjugate of Theorem 2.2); but it is only for symmetric spinors which are restricted to only one type of index where we can *always* obtain a *symmetric* potential.

3. Potentials for symmetric (3,1)-spinors in Einstein space–times

3.1. Introduction

In some special cases [1,3,5,9,27] there has been found a completely symmetric spinor $H_{ABA'B'}$ such that the spinor

$$L_{ABCA'} = \nabla_{(A}{}^{B'}H_{BC)A'B'}$$

is a Lanczos potential of the Weyl spinor. In this section, we will prove that such a spinor $H_{ABA'B'}$ exists in all Einstein space–times, i.e., space–times such that the Ricci spinor $\Phi_{ABA'B'} = 0$. In fact, we will prove that in such space–times *any* symmetric spinor $L_{ABCA'}$ can be written as

$$L_{ABCA'} = \nabla_{(A}{}^{B'}H_{BC)A'B'}$$

for some spinor $H_{ABA'B'} = H_{(AB)(A'B')}$ and that for each choice of $L_{ABCA'}$ there exists many such spinors $H_{ABA'B'}$. We emphasize that this result does *not* follow from Theorem 2.2 because here we are requiring *complete* symmetry of $H_{ABA'B'}$.

3.2. A preliminary result

First we need a preliminary lemma, which is of interest in its own right.

Lemma 3.1. *For any symmetric spinor field φ_{AB}, time-like or space-like vector field $n^{AA'}$ and complex function f there exists a unique complex vector field $\zeta^{AA'}$ such that $\varphi_{BC} = n_{(B}{}^{A'}\zeta_{C)A'}$ and in addition $n^{AA'}\zeta_{AA'} = f$.*

Proof. By rescaling, it suffices to assume that $n^{AA'}$ is a unit time-like or space-like vector. We start by proving uniqueness; suppose that $\varphi_{BC} = n_{(B}{}^{A'}\zeta_{C)A'}$ and $n^{AA'}\zeta_{AA'} = f$. For the case when $n^{AA'}$ is time-like we obtain

$$\begin{aligned} 2\varphi_A{}^B n_{BA'} + fn_{AA'} &= n_A{}^{B'}\zeta^B_{B'}n_{BA'} + n^{BB'}\zeta_{AB'}n_{BA'} + fn_{AA'} \\ &= \tfrac{1}{2}\zeta_{AA'} + n_{AA'}n_B{}^{B'}\zeta_{B'}{}^B + \varepsilon_{BA}n_{CA'}n^{CB'}\zeta_{B'}{}^B + fn_{AA'} = \zeta_{AA'}, \end{aligned} \quad (4)$$

where we have used that $n_{B'}{}^C n_{CA'} = \frac{1}{2}\varepsilon_{A'B'}$. In the space-like case, the same calculations give

$$2\varphi_A{}^B n_{BA'} - fn_{AA'} = \zeta_{AA'}.$$

This proves the uniqueness part so now we need only verify that the above candidate for $\zeta_{AA'}$ actually satisfies the conclusion of the lemma. As before we start with the time-like case

$$n^{AA'}(2\varphi_A{}^B n_{BA'} + fn_{AA'}) = 2\varphi_A{}^B \cdot \tfrac{1}{2}\varepsilon_B{}^A + f = f$$

and

$$n_{(B}{}^{A'}\zeta_{C)A'} = 2n_{(B}{}^{A'}\varphi_{C)}{}^D n_{DA'} + fn_{(B}{}^{A'}n_{C)A'} = \varepsilon_{D(B}\varphi_{C)}{}^D = \varphi_{BC}.$$

This proves the lemma in the time-like case. The space-like case is proved in exactly the same way. □

3.3. Construction of the spinor potential

Let M be an Einstein space–time, i.e., $\Phi_{ABA'B'} = 0$ and let $L_{ABCA'}$ be a symmetric spinor field on M. Our objective is to show that locally there exists a spinor field $H_{ABA'B'} = H_{(AB)(A'B')}$ such that

$$L_{ABCA'} = \nabla_{(A}{}^{B'}H_{BC)A'B'}. \quad (5)$$

We also wish to examine the gauge freedom in the potential $H_{ABA'B'}$.

Our strategy for proving the existence of $H_{ABA'B'}$ will be to start by deriving a wave equation for $H_{ABA'B'}$, along with some constraint equations. Then, we use a theorem from [15] to show that these equations have a solution; finally, we prove that this solution also solves Eq. (5).

We begin by assuming that $H_{ABA'B'} = H_{(AB)(A'B')}$ satisfies

$$\nabla_{(A}{}^{B'} H_{BC)A'B'} = L_{ABCA'}, \qquad \nabla^{AA'} H_{ABA'B'} = \zeta_{BB'}, \tag{6}$$

where $\zeta_{BB'}$ is a given spinor field (complex 1-form). Note that

$$\nabla_A{}^{B'} H_{BCA'B'} = \nabla_{(A}{}^{B'} H_{BC)A'B'} - \tfrac{2}{3}\varepsilon_{A(B}\nabla^{DD'} H_{C)DA'D'},$$

so (6) is equivalent to

$$\nabla_A{}^{B'} H_{BCA'B'} = L_{ABCA'} - \tfrac{2}{3}\varepsilon_{A(B}\zeta_{C)A'}. \tag{7}$$

Now, let Σ be a C^∞ space-like past-compact hypersurface with future-directed unit normal $n^a = n^{AA'}$. Let $\nabla_n = n^a\nabla_a = n^{AA'}\nabla_{AA'}$ be the normal derivative with respect to Σ and let $\tilde{\nabla}_{AA'} = \nabla_{AA'} - n_{AA'}\nabla_n$. We remark that on Σ, $\tilde{\nabla}_{AA'}T$, where T is an arbitrary spinor field, depends only on the restriction of T to Σ. This is most easily shown using Gaussian normal coordinates [29]. We will therefore, by some abuse of notation, allow $\tilde{\nabla}_{AA'}$ to act on spinor fields defined only on Σ.

Put $\mathring{H}_{ABA'B'} = H_{ABA'B'}|_\Sigma$. Since (7) must be satisfied also on Σ, we obtain

$$n_A{}^{B'}\nabla_n H_{BCA'B'}|_\Sigma = -(\tilde{\nabla}_A{}^{B'} H_{BCA'B'} - L_{ABCA'} + \tfrac{2}{3}\varepsilon_{A(B}\zeta_{C)A'})|_\Sigma.$$

As before, we have that $n^{AC'}n_A{}^{B'} = \frac{1}{2}\varepsilon^{B'C'}$. Thus, multiplying the previous equation by $n^{AC'}$ gives us an explicit expression for the normal derivative of $H_{ABA'B'}$.

$$\nabla_n H_{BCA'}{}^{C'}|_\Sigma = 2n^{AC'}(\tilde{\nabla}_A{}^{B'} H_{BCA'B'} - L_{ABCA'} + \tfrac{2}{3}\varepsilon_{A(B}\zeta_{C)A'})|_\Sigma. \tag{8}$$

Note that since $\tilde{\nabla}_A{}^{B'} H_{BCA'B'}$ depends only on the restriction of H to Σ, we can replace H with $\mathring{H}$ in the RHS. If we lower the index C' then the LHS is symmetric over $(A'C')$. Hence, the above equation is equivalent to the following initial value constraints:

$$\begin{aligned} &\nabla_n H_{BCA'C'}|_\Sigma = 2n^A_{(C'}(\tilde{\nabla}_{|A}{}^{B'}\mathring{H}_{BC|A')B'} - L_{|ABC|A')} + \tfrac{2}{3}\varepsilon_{|A(B}\mathring{\zeta}_{C)|A')})|_\Sigma, \\ &0 = n^{AA'}(\tilde{\nabla}_A{}^{B'}\mathring{H}_{BCA'B'} - L_{ABCA'} + \tfrac{2}{3}\varepsilon_{A(B}\mathring{\zeta}_{C)A'})|_\Sigma, \end{aligned} \tag{9}$$

where we have put $\mathring{\zeta}_{AA'} = \zeta_{AA'}|_\Sigma$.

Next, we differentiate the LHS of (7)

$$\begin{aligned} \nabla^A_{C'}\nabla_A{}^{B'} H_{BCA'B'} &= \varepsilon^{B'D'}\nabla^A_{(C'}\nabla_{D')A} H_{BCA'B'} + \tfrac{1}{2}\nabla^A_{E'}\nabla_A{}^{E'} H_{BCA'C'} \\ &= -\tfrac{1}{2}\Box H_{BCA'C'} + \bar{\Psi}_{B'E'A'C'} H_{BC}{}^{E'B'} - 4\Lambda H_{BCA'C'}, \end{aligned} \tag{10}$$

where we have used that $\Phi_{ABA'B'} = 0$ along with the symmetry of $H_{ABA'B'}$. Thus, $H_{ABA'B'}$

satisfies the following wave equation:

$$\Box H_{BCA'C'} - 2\bar{\Psi}_{B'E'A'C'} H_{BC}{}^{E'B'} + 8\Lambda H_{BCA'C'} \\ = -2\nabla^{A}_{C'} L_{ABCA'} + \tfrac{4}{3}\nabla_{C'(B}\zeta_{C)A'}. \tag{11}$$

Note that this equation is actually a special case of Eq. (2).

Since, $H_{BCA'C'}$ is symmetric over $(A'C')$, it follows that (11) is equivalent to

$$\Box H_{BCA'C'} - 2\bar{\Psi}_{B'E'A'C'} H_{BC}{}^{E'B'} + 8\Lambda H_{BCA'C'} \\ = -2\nabla^{A}_{(C'} L_{|ABC|A')} + \tfrac{2}{3}\nabla_{C'(B}\zeta_{C)A'} + \tfrac{2}{3}\nabla_{A'(B}\zeta_{C)C'}, \\ 0 = -\nabla^{AA'} L_{ABCA'} + \tfrac{2}{3}\nabla_{(B}{}^{A'}\zeta_{C)A'}. \tag{12}$$

The second of these equations is actually Eq. (3).

After these preliminary considerations, we are ready to prove our main result.

Theorem 3.2. *Suppose M is an Einstein space–time ($\Phi_{ABA'B'} = 0$) and that $\Sigma \subset M$ is a C^∞ space-like past-compact hypersurface with future directed unit normal $n^{AA'}$. Let a spinor field $L_{ABCA'} = L_{(ABC)A'}$ and a complex function g be given. Furthermore, let a spinor field $\mathring{\mathrm{H}}_{ABA'B'} = \mathring{\mathrm{H}}_{(AB)(A'B')}$ and a complex function $\mathring{\mathrm{f}}$, both defined only on Σ, be given. Then, there exists a neighbourhood U of Σ such that there exists a unique spinor field $H_{ABA'B'} = H_{(AB)(A'B')}$ satisfying the equations*

$$\nabla_{(A}{}^{B'} H_{BC)A'B'} = L_{ABCA'}, \qquad \nabla^{AA'}\nabla^{BB'} H_{ABA'B'} = g, \\ H_{ABA'B'}|_\Sigma = \mathring{\mathrm{H}}_{ABA'B'}, \qquad n^{AA'}\nabla^{BB'} H_{ABA'B'}|_\Sigma = \mathring{\mathrm{f}}, \tag{13}$$

on all of U.

Proof. An outline of the existence part of the proof is as follows. We start by solving the second of Eq. (9) for $\mathring{\zeta}_{AA'}$ so that $n^{AA'}\mathring{\zeta}_{AA'} = \mathring{\mathrm{f}}$. Then, we evolve this initial data using the second equation of (12) in such a way that $\nabla^{AA'}\zeta_{AA'} = g$. Next, we calculate the normal derivative of $H_{ABA'B'}$ using the first equation of (9) and use the so obtained Cauchy data for $H_{ABA'B'}$ to solve the first equation of (12) for $H_{ABA'B'}$. It can then be verified that this spinor field satisfies all the conditions of the theorem.

Define the symmetric spinor

$$\mathring{\varphi}_{BC} = \tfrac{3}{2} n^{AA'}(L_{ABCA'}|_\Sigma - \tilde{\nabla}_A{}^{B'}\mathring{\mathrm{H}}_{BCA'B'}).$$

By Lemma 3.1, there exists a unique spinor $\mathring{\zeta}_{AA'}$ such that

$$n^{AA'}\mathring{\zeta}_{AA'} = \mathring{\mathrm{f}},$$

and such that the second of Eq. (9) is satisfied, i.e.,

$$\mathring{\varphi}_{BC} = n_{(B}{}^{A'}\mathring{\zeta}_{C)A'}. \tag{14}$$

Our next task will be to solve for $\zeta_{AA'}$. We want to find $\zeta_{AA'}$, so that the following three conditions are satisfied:

$$\nabla_{(B}{}^{A'}\zeta_{C)A'} = \tfrac{3}{2}\nabla^{AA'} L_{ABCA'}, \qquad \nabla^{AA'}\zeta_{AA'} = g, \qquad \zeta_{AA'}|_\Sigma = \mathring{\zeta}_{AA'}, \tag{15}$$

where $\mathring{\zeta}_{AA'}$ is the solution of (14) obtained above. Let U be a causal neighbourhood [15] of Σ. According to Theorem 2.1, this problem has a unique solution $\zeta_{AA'}$ in U.

Next, consider the problem

$$\begin{aligned}&\Box H_{BCA'C'} - 2\bar{\Psi}_{B'E'A'C'} H_{BC}{}^{E'B'} + 8\Lambda H_{BCA'C'}\\&\quad = -2\nabla^{A}_{(C'} L_{|ABC|A')} + \tfrac{2}{3}\nabla_{C'(B}\zeta_{C)A'} + \tfrac{2}{3}\nabla_{A'(B}\zeta_{C)C'},\\&\nabla_n H_{BCA'C'}|_\Sigma = 2n^{A}_{(C'}(\tilde{\nabla}_{|A}{}^{B'}\mathring{H}_{BC|A')B'} - L_{|ABC|A')} + \tfrac{2}{3}\varepsilon_{|A(B}\mathring{\zeta}_{C)|A')})|_\Sigma,\\&H_{BCA'C'}|_\Sigma = \mathring{H}_{BCA'C'}.\end{aligned} \tag{16}$$

These are the first equation of (12), the first equation of (9) and the third condition of (13). Note that the RHS of all three equations contain only known quantities. Hence, this problem is a Cauchy problem for a linear, diagonal, second-order hyperbolic system. According to a theorem in [15,29] this problem has a unique solution $H_{BCA'C'}$ in U.

It now remains to prove that the $H_{BCA'C'}$ found above satisfies the conditions

$$\nabla_{(A}{}^{B'} H_{BC)A'B'} = L_{ABCA'}, \qquad \nabla^{BB'} H_{ABA'B'} = \zeta_{AA'}.$$

In order to do that, we define

$$\xi_{ABCA'} = \nabla_A{}^{B'} H_{BCA'B'} - L_{ABCA'} + \tfrac{2}{3}\varepsilon_{A(B}\zeta_{C)A'}.$$

Eq. (9) now implies that $\xi_{ABCA'}|_\Sigma = 0$. Because both $H_{BCA'C'}$ and $\zeta_{AA'}$ are constructed so that Eq. (12) are satisfied, we have that $\nabla^{A}_{C'}\xi_{ABCA'} = 0$. Because $(\tilde{\nabla}^{A}_{C'}\xi_{ABCA'})|_\Sigma$ only depends on $\xi_{ABCA'}|_\Sigma$, this gives us that

$$n^{A}_{C'}\nabla_n \xi_{ABCA'}|_\Sigma = -(\tilde{\nabla}^{A}_{C'}\xi_{ABCA'})|_\Sigma = 0.$$

Thus,

$$0 = n^{DC'} n^{A}_{C'}\nabla_n \xi_{ABCA'}|_\Sigma = \tfrac{1}{2}\varepsilon^{AD}\nabla_n\xi_{ABCA'}|_\Sigma = -\tfrac{1}{2}\nabla_n \xi^{D}{}_{BCA'}.$$

Taking another derivative gives us

$$\begin{aligned}0 &= \nabla_D{}^{C'}\nabla^{A}_{C'}\xi_{ABCA'} = -\tfrac{1}{2}\Box\xi_{DBCA'} + 2\Psi_{D(B}{}^{AF}\xi_{|A|C)FA'} - 3\Lambda\xi_{DBCA'}\\&\quad -2\Lambda\xi_{(BC)DA'} - 2\Lambda\varepsilon_{D(B}\xi^{A}{}_{C)AA'},\end{aligned} \tag{17}$$

because we assumed that M is Einstein. Hence, $\xi_{ABCA'}$ is a solution of the following problem:

$$\begin{aligned}&\Box\xi_{DBCA'} - 4\Psi_{D(B}{}^{AF}\xi_{|A|C)FA'} + 6\Lambda\xi_{DBCA'} + 4\Lambda\xi_{(BC)DA'} + 4\Lambda\varepsilon_{D(B}\xi^{A}_{C)AA'} = 0,\\&\xi_{DBCA'}|_\Sigma = 0, \qquad \nabla_n\xi_{DBCA'}|_\Sigma = 0\end{aligned} \tag{18}$$

This homogeneous problem has a unique solution in U according to [15]. Therefore, we must have

$$\xi_{ABCA'} = 0$$

in U, which implies that

$$\nabla_{(A}{}^{B'} H_{BC)A'B'} = L_{ABCA'}, \qquad \nabla^{BB'} H_{ABA'B'} = \zeta_{AA'}.$$

This proves that $H_{BCA'C'}$ satisfies all the conditions (13), which completes the existence part of the theorem.

Uniqueness. Remember that $\mathring{\zeta}_{AA'}$ was uniquely determined by the fourth condition of (13) and the second equation of (9) and that $\zeta_{AA'}$ was uniquely determined by $\mathring{\zeta}_{AA'}$, the second condition of (13) and the second equation of (12). Also recall that this determined the normal derivative of $H_{BCA'C'}$ on Σ uniquely and that this normal derivative together with the third condition of (13) and the first equation of (12) determined $H_{BCA'C'}$ uniquely. This establishes uniqueness. □

3.4. The tensor potential

It is tedious but straightforward to translate the above result into tensors. The condition $\Phi_{ABA'B'} = 0$ translates into the vanishing of the trace-free Ricci tensor $\tilde{R}_{ab} = R_{ab} - \frac{1}{4}Rg_{ab}$ and as mentioned above, $L_{ABCA'}$ corresponds to a real tensor L_{abc} such that

$$L_{abc} = L_{[ab]c}, \qquad L_{[abc]} = 0, \qquad L_{ab}{}^{b} = 0.$$

We also note that a spinor field $H_{ABA'B'} = H_{(AB)(A'B')}$ corresponds to a complex, symmetric and trace-free tensor field H_{ab}, i.e.,

$$H_{ab} = H_{(ab)}, \qquad H_a{}^{a} = 0.$$

Theorem 3.3. *Suppose M is an Einstein space–time* ($\tilde{R}_{ab} = 0$) *and that* $\Sigma \subset M$ *is a* C^∞ *space-like past-compact hypersurface with future directed unit normal* n^a. *Let a real tensor field* L_{abc} *having the above symmetries, and a complex function g be given. Furthermore, let a complex function* $\mathring{\mathrm{f}}$ *and a complex tensor field* $\mathring{\mathrm{H}}_{ab} = \mathring{\mathrm{H}}_{(ab)}$ *such that* $\mathring{\mathrm{H}}_a{}^{a} = 0$, *both defined only on* Σ *be given. Then, there exists a neighbourhood U of* Σ *such that there exists a unique complex tensor field* H_{ab} *satisfying the equations*

$$\begin{aligned} &H_{ab} = H_{(ab)}, \qquad H_a{}^{a} = 0, \\ &L_{abc} = -\nabla_{[a}H_{b]c} - \nabla_{[a}\bar{H}_{b]c} - \mathrm{i}\nabla^*_{[a}H_{b]c} + \mathrm{i}\nabla^*_{[a}\bar{H}_{b]c} + \tfrac{1}{3}(g_{c[a}\nabla^d H_{b]d} \\ &\qquad + g_{c[a}\nabla^d \bar{H}_{b]d} + \mathrm{i}g^*_{c[a}\nabla^d H_{b]d} - \mathrm{i}g^*_{c[a}\nabla^d \bar{H}_{b]d}), \\ &\nabla^a\nabla^b H_{ab} = g, \qquad H_{ab}|_\Sigma = \mathring{\mathrm{H}}_{ab}, \qquad n^a\nabla^b H_{ab}|_\Sigma = \mathring{\mathrm{f}}, \end{aligned} \tag{19}$$

on all of U.

By writing $H_{ab} = H^1_{ab} + \mathrm{i}H^2_{ab}$, where H^1 and H^2 are real, we can simplify the second of the above conditions somewhat

$$L_{abc} = -2\nabla_{[a}H^1_{b]c} + 2\nabla^*_{[a}H^2_{b]c} + \tfrac{2}{3}(g_{c[a}\nabla^d H^1_{b]d} - g^*_{c[a}\nabla^d H^2_{b]d}). \tag{20}$$

4. Comparison with electromagnetic theory

4.1. Introduction

In this section, we consider electromagnetic theory in a curved space–time. Most of the above results are applicable here too, and we will also find that due to the simple index configuration of the electromagnetic spinor (φ_{AB} as compared to Ψ_{ABCD}) and also due to Maxwell's equations, certain simplifications will occur.

4.2. The electromagnetic field and its spinor potentials

First of all we remark that as in the rest of the paper all results in this section are local in nature unless comments are made to the contrary.

Recall that the electromagnetic tensor (Maxwell tensor) is a 2-form $F_{ab} = F_{[ab]}$. Maxwell's equations are

$$\nabla^a F_{ab} = J_b, \qquad \nabla_{[a} F_{bc]} = 0,$$

where J_b is the source current. The second of these equations together with Poincare's lemma gives us the existence of a (real) 1-form A_a such that

$$F_{ab} = \nabla_{[a} A_{b]}.$$

Now, put $\alpha = \nabla^a A_a$ (i.e., α is analogous to the differential gauge in the above sections). If $\alpha = 0$, the electromagnetic potential A_a is said to be in Lorentz gauge.

To examine the gauge freedom in A_a, suppose A_a and $\tilde{A}_a$ are two potentials of F_{ab} in the same differential gauge and put $B_a = \tilde{A}_a - A_a$. Then

$$\nabla_{[a} B_{b]} = 0, \qquad \nabla^a B_a = 0. \tag{21}$$

Thus, there exists a (real) scalar field G such that $B_a = \nabla_a G$ and by the second condition, then $\Box G = 0$.

Conversely, take any scalar field G that satisfies $\Box G = 0$ and put $B_a = \nabla_a G$. Then, B_a satisfies Eq. (21) and therefore $\tilde{A}_a = A_a + B_a$ will be a potential of F_{ab} in the same differential gauge as A_a. Hence, we have completely characterized the gauge transformations that preserve the differential gauge.

Next, we turn to the spinor formulation. As F_{ab} is antisymmetric, it can be written

$$F_{ab} = \varphi_{AB}\varepsilon_{A'B'} + \bar{\varphi}_{A'B'}\varepsilon_{AB}$$

for some symmetric spinor φ_{AB}. Maxwell's equations can be shown to be

$$\nabla_{A'}{}^{B}\varphi_{AB} = J_{AA'},$$

where $J_{AA'}$ is the hermitian spinor equivalent of the current J_a. If we apply Illge's Theorem 2.2 to φ_{AB}, we obtain the existence of a *complex* 1-form $A_{AA'}$ such that

$$\varphi_{AB} = \nabla_{(A}{}^{A'} A_{B)A'}. \tag{22}$$

Putting $A_{AA'} = -\frac{1}{2}A^1_{AA'} + \frac{1}{2}\mathrm{i}A^2_{AA'}$, where $A^i_{AA'}, i = 1, 2$ are hermitian, this equation becomes (in tensors) [18]

$$F_{ab} = \nabla_{[a}A^1_{b]} + {}^*\nabla_{[a}A^2_{b]},$$

where * denotes the Hodge dual. It is shown in [18] that solutions of this equation with $A^2_a = 0$ exist if and only if $\nabla_{[a}F_{bc]} = 0$ (which is true if and only if $J_{AA'}$ is hermitian) in agreement with Poincare's lemma.

It is interesting to note that the existence of the potential A_a in electromagnetic theory is usually presented as a consequence of the second of Maxwell's equations via Poincare's lemma. However, we see that the existence of the (complex) potential $A_{AA'}$ is independent of Maxwell's equations; it is simply a consequence of Theorem 2.2. The role of Maxwell's equations is to ensure that this potential can be chosen hermitian.

Now, we can of course use the theorems in the earlier sections to find potentials of $A_{AA'}$. From Theorem 2.2, we know that we can always find an asymmetric potential $H_{A'B'}$ (however, when $A_{AA'}$ is divergence-free, i.e., $\alpha = 0$ it is shown in [18] that a symmetric potential always exists, see also below) and from the complex conjugate of Theorem 2.2 (or 3.3), we can obtain an asymmetric potential T_{AB}. So, we have two potentials for $A_{AA'}$ satisfying

$$A_{AA'} = \nabla_{A'}{}^{B}T_{AB} = \nabla_{A}{}^{B'}H_{A'B'}.$$

It is easily seen that if $A_{AA'}$ is hermitian and if T_{AB} is a potential of $A_{AA'}$, then $H_{A'B'} = \bar{T}_{A'B'}$ is also a potential of $A_{AA'}$.

It is to be noted that if F_{ab} does not satisfy Maxwell's equations then we cannot choose the electromagnetic potential $A_{AA'}$ hermitian, and there is no simple relation between the two potentials T_{AB} and $H_{A'B'}$.

As before we can also obtain a wave equation for T_{AB}. Decomposed into its symmetric and antisymmetric parts it becomes

$$0 = \Box T_{(AB)} - 2\Psi_{AB}{}^{CD}T_{(CD)} + 8\Lambda T_{(AB)} + 2\varphi_{AB}, \qquad 0 = \Box T_{A}{}^{A} + 2\alpha, \tag{23}$$

highlighting a formal resemblance between T_{AB} and the Hertz potential in flat space.

We can express the gauge freedom of $A_{AA'}$ in terms of T_{AB}. The result is that $A_{AA'}$ and $\tilde{A}_{AA'} = A_{AA'} + B_{AA'}$ are two potentials of φ_{AB} in the differential gauge α if and only if

$$B_{AA'} = \nabla_{A'}{}^{B}T_{AB},$$

where T_{AB} is a solution of

$$0 = \Box T_{(AB)} - 2\Psi_{AB}{}^{CD}T_{(CD)} + 8\Lambda T_{(AB)}, \qquad 0 = \Box T_{A}{}^{A}. \tag{24}$$

But we had already expressed the gauge freedom in terms of the scalar G for the Maxwell case when $B_{AA'}$ is hermitian, so we might wonder what the link between T_{AB} and G is. To give a partial answer to this question, let Σ be as in Section 5 and suppose

$$B_{AA'} = \nabla_{A'}{}^{B}T_{AB} = \nabla_{AA'}G,$$

where T_{AB} satisfies the first of Eq. (24) and G is an arbitrary real scalar field (so that the gauge transformation $B_{AA'}$ is allowed to change the differential gauge). It follows that

$$0 = \nabla_{A'}{}^{B}(T_{AB} + \varepsilon_{AB}G).$$

By differentiating again, we obtain ($S_{AB} = T_{(AB)}$, $T = T_{A}{}^{A}$)

$$0 = \Box S_{AB} - 2\Psi_{AB}{}^{CD}S_{CD} + 8\Lambda S_{AB}, \qquad 0 = \Box(T + 2G), \tag{25}$$

and by evaluating on Σ, we get

$$\nabla_n S_{AC}|_{\Sigma} = (-2n_{A'(C}\tilde{\nabla}^{A'B}S_{A)B} + n_{A'(C}\tilde{\nabla}^{A'}_{A)}T)|_{\Sigma},$$
$$\nabla_n(T + 2G)|_{\Sigma} = 2n^{AA'}\tilde{\nabla}_{A'}{}^{B}S_{AB}|_{\Sigma}. \tag{26}$$

It easily follows that if $T|_{\Sigma} = -2G|_{\Sigma}$ and if $n^{AA'}\tilde{\nabla}_{A'}{}^{B}S_{AB}|_{\Sigma} = 0$, then $T = -2G$ in a neighbourhood of Σ.

Finally, we will look a little closer at the case when F_{ab} is a 2-form that satisfies Maxwell's equations. Poincare's lemma (or [18]) then tells us that there exists a (hermitian) divergence-free potential $A_a = A_{AA'}$. Now, according to Illge [18] for any *complex* divergence-free 1-form $A_{AA'}$ there exists a *symmetric* spinor T_{AB} such that $A_{AA'} = \nabla_{A'}{}^{B}T_{AB}$. Define the 2-form $T_{ab} = T_{AB}\varepsilon_{A'B'} + \bar{T}_{A'B'}\varepsilon_{AB}$. The tensor equations relating A_a and T_{ab} are then

$$\nabla^{a}T_{ab} = 2\,\mathrm{Re}(A_a), \qquad {}^{*}\nabla^{a}T_{ab} = 2\,\mathrm{Im}(A_a).$$

As A_a was chosen hermitian, we obtain

$$\nabla^{a}T_{ab} = 2A_a, \qquad {}^{*}\nabla^{a}T_{ab} = 0.$$

The second equation of these is equivalent to $\nabla_{[a}T_{bc]}$, i.e., T_{ab} is a closed 2-form just like F_{ab} so T_{ab} also has a hermitian, divergence-free potential and so on. Hence, we get an infinite chain of potentials alternating between hermitian, divergence-free 1-forms and closed hermitian 2-forms.

5. Discussion

The most important motivation for studying the general spinor potentials of the earlier sections has been the Lanczos potential of the Weyl curvature spinor. The discussion of this section will therefore deal mainly with those potentials and their 'superpotentials' $H_{ABA'B'}$ and T_{ABCD} from [6].

Due no doubt in part to the rather complicated tensor version (1) of its relationship to the Weyl tensor, and also to various mistakes in some papers, the Lanczos potential has failed to attract major attention, and there is perhaps still an air of uncertainty surrounding it.

Although Bampi and Caviglia [7] identified the flaw in Lanczos' original attempt to prove its existence, the complicated nature of their own existence proof also helped to set the Lanczos potential apart. Maher and Zund [23] discovered the very simple and natural spinor structure of $L_{ABCA'}$ as early as 1968. This result attracted little interest however, perhaps because of some misprints and ambiguities in this and subsequent papers.

Twenty years later, Illge's work [18] highlighted and exploited the spinor representation, and also discovered for the first time the remarkably simple wave equation for the Lanczos potential of the Weyl spinor in vacuum space–times and Lanczos differential gauge. (Although Lanczos had calculated a wave equation for the Lanczos potential of the Weyl tensor in tensor notation, containing complicated non-linear terms obtained by everywhere replacing C_{abcd} with the appropriate expression in L_{abc}, it contained some mistakes, which were repeated, or only partly corrected by others; no-one had suspected that these non-linear terms were actually identically zero in four dimensions.) The relative simplicity of the Lanczos spinor wave equation in the less ideal cases of non-vacuum, arbitrary differential gauge, arbitrary W_{ABCD} (in particular it is linear) enabled Illge to use the wave equation in his proof of existence of the Lanczos potential. More precisely he showed the equivalence of the solution set of the wave equation subject to an initial value constraint, and the solution set of the Weyl–Lanczos equation.

By applying the spinor results of Illge [18] and this paper to electromagnetic theory in Section 4 we emphasized, as pointed out by Illge [18] that the existence of the electromagnetic potential is not dependent on the second of Maxwell's equations, via Poincare's lemma, which is the way in which it is usually presented. The simplification of having a hermitian potential $A_{AA'}$ can clearly not apply to $L_{ABCA'}$ or its potential T_{ABCD}, however, such a possibility may exist for the potential $H_{ABA'B'}$ of $L_{ABCA'}$ (for, at least, a significant class of space–times), and this is one of the questions requiring further investigations. We also saw that in the case of a hermitian $A_{AA'}$ the two potentials T_{AB} and $H_{A'B'}$ were essentially equivalent; whether there are any (less direct) links between the two potentials of $L_{ABCA'}$ also requires further investigations.

The existence of a potential such as $L_{ABCA'}$ for Ψ_{ABCD} is of course well known and thoroughly investigated in flat space in connection with the massless field equation; and indeed a chain of Hertz-like potentials, including some analogous to T_{ABCD} and $H_{ABA'B'}$, have been studied. Although Penrose [26] has studied these using spinor techniques, his results are strictly for (conformally) flat space–times. In $\mathcal{H}$-spaces (complex general relativity) [19] the complex connection plays the role of a complex Lanczos potential $L_{ABCA'}$ of one of the Weyl spinors (recall that the other Weyl spinor is zero since $\mathcal{H}$-spaces are always left-flat), and this potential itself *always* permits a potential $H_{ABA'B'}$. This H-potential is the basis for constructing physics in $\mathcal{H}$-spaces. This is part of our motivation for investigating, in Section 5, the existence of an H-potential in real curved space. It is hoped, having now shown that such a potential does exist in physically important curved spaces, that (at least part of) the successful programme associated with the complex H-potential can be applied to this H-potential in real space–times.

A related motivation is that in earlier investigations of Lanczos potentials, the existence of such an $H_{ABA'B'}$ was not only an important aid to calculate the Lanczos potential $L_{ABCA'}$, but the possibility of it having physical and geometrical significance has also been considered. We summarize those cases below:

- Torres del Castillo [27,28] has studied space–times admitting a normalized spinor dyad (o^A, ι^A) in which

  $$\kappa = \sigma = 0,$$

  and in which the Ricci spinor satisfies

  $$\Phi_{ABA'B'}o^A o^B = 0.$$

  He found that in all such spaces there exists a Lanczos potential $L_{ABCA'}$ of the Weyl spinor such that $L_{ABCA'}$ can be written

  $$L_{ABCA'} = \nabla_{(A}{}^{B'} H_{BC)A'B'}$$

  for some completely symmetric spinor $H_{ABA'B'} = Q_{AB}o_{A'}o_{B'}$ (we remark that for the case $\rho \neq 0$, $\Lambda =$ constant, all such Lanczos- and H-potentials have been calculated [3] using the GHP-formalism). By defining

  $$\eta_{ab} = g_{ab} - H_{ab},$$

  where H_{ab} is the symmetric, trace-free tensor equivalent of $H_{ABA'B'}$ Torres del Castillo obtained a complex, conformally flat metric η_{ab}.
- Bergqvist and Ludvigsen [10] define a curvature-free connection in the Kerr space–time, by

  $$\hat{\nabla}_{AA'}\xi^B = \nabla_{AA'}\xi^B + 2\Gamma_C{}^B{}_{AA'}\xi^C,$$

  where

  $$\Gamma_{ABCA'} = \nabla_{(A}{}^{B'} H_{B)CA'B'}, \tag{27}$$

  and $H_{ABA'B'}$ is hermitian and given by

  $$H_{ABA'B'} = \frac{\rho + \bar{\rho}}{4\rho^3}\Psi_2 o_A o_B o_{A'} o_{B'}, \tag{28}$$

  where o^A is a repeated principal spinor of the Weyl spinor. Subsequently, Bergqvist [9] has shown that $\Gamma_{(ABC)A'}$ is a Lanczos potential in the Kerr space–time. This connection has been used by Bergqvist and Ludvigsen [10] to construct quasi-local momentum in the Kerr space–time.
- In [5], these results are generalized to Kerr–Schild space–times, i.e.,

  $$g_{ab} = \eta_{ab} + fl_a l_b,$$

  where η_{ab} is a flat metric, l^a is null and f is a real function. It is shown that providing l^a is geodesic and shear-free (or if another more technical condition is fulfilled) then $H_{ab} = fl_a l_b$ is a hermitian H-potential of a Lanczos potential of the Weyl spinor, that also defines a curvature-free connection (see also [16]).

- These results are further generalized in [2,3]. It is shown that in a class of spaces admitting a geodesic shear-free expanding null congruence (including all vacuum ones) an H-potential of a Lanczos potential of the Weyl spinor such that it defines a completely curvature-free connection, can always be found.
- In a recent paper López-Bonilla et al. [21] have found, for the Kerr space–time, an explicit Lanczos potential of the Weyl spinor, given by a hermitian H-potential of the type discussed in this paper, for the Kerr space–time. Note that this H-potential is different from the one found by Bergqvist and Ludvigsen [10].
- Novello and Velloso [24] have shown that for perfect fluid space–times that admit a normalized time-like vector field u^a, $u_a u^a = 1$ which is shear-free and vorticity-free, so that

  $$\nabla_a u_b = u_a \dot{u}_b + \tfrac{1}{3}\theta h_{ab},$$

  where $\dot{u}_a = u^b \nabla_b u_a$, $h_{ab} = g_{ab} - u_a u_b$ and θ is the expansion of u^a, then the tensor

  $$L_{abc} = 2\dot{u}_{[a} u_{b]} u_c + \tfrac{2}{3} g_{c[a} \dot{u}_{b]} \tag{29}$$

  is a Lanczos potential of the Weyl spinor (the second term is to ensure that ${L_{ab}}^b = 0$). It is easy to confirm that when

  $$H_{ab} = u_a u_b - \tfrac{1}{4} g_{ab} = \tfrac{3}{4} u_a u_b - \tfrac{1}{4} h_{ab} \tag{30}$$

  is substituted for H^1 (with $H^2 = 0$) in Eq. (20), we obtain precisely the Lanczos potential (29).

We conclude with two comments. In some of the examples quoted above an H-potential of a Lanczos potential of the Weyl spinor was found for some non-Einstein space–times; it remains an open question if such a construction is possible for a significant class of non-Einstein space–times. Earlier in this section, we commented on the possible significance of having a hermitian H-potential. Another open question is whether hermitian superpotentials for the Weyl spinor can be found for a more general class of space–times.

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